HEAT TRANSFER AND RESISTANCE OF AN INCOMPRESSIBLE LIQUID IN THE INITIAL PORTION OF A CIRCULAR TUBE FOR VARIOUS LAWS OF HEAT SUPPLY

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We consider heat transfer for the case of laminar flow of an incompressible liquid in the initial portion of a circular tube with a parabolic velocity profile at its entrance. Analytical expressions are obtained for the dependence of the Nusselt number Nu and the resistance coefficient ξRe on the longitudinal coordinate for various laws of heat transfer.

For the case in which the heat supply along a tube is a function of the longitudinal coordinate the existing methods of calculating the local Nusselt number are laborious and require a considerable number of manipulations. In this paper we use an approximate analytical method to solve the problem of the laminar flow of an incompressible liquid in the initial portion of a circular tube with a parabolic velocity profile at its entrance; the heat supply along the tube wall is assumed to be arbitrary; we also solve the problem of liquid flow in the initial portion of a thermally insulated tube where hydrodynamic stabilization at the tube entrance is absent.

We obtain expressions for the Nusselt number and the resistance coefficient acceptable for engineering calculations. Consider the flow of an incompressible liquid in the initial portion of a circular tube. We assume the liquid to have constant physical properties and that its temperature profile at the tube entrance is constant over a given cross section.

In the presence of hydrodynamic stabilization the velocity profile as a function of the radius follows Poiseuille's law:

$$u = 2\bar{u}\left[1 - 4\left(\frac{r}{d}\right)^2\right],$$

and the energy equation has the form

$$2\rho c_p \bar{u}\left[1 - 4\left(\frac{r}{d}\right)^2\right]\frac{\partial T}{\partial x} = \lambda\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

with the boundary and initial conditions

$$\frac{\partial T}{\partial r} = 0 \quad \text{for} \quad r = 0,$$

$$\lambda\frac{\partial T}{\partial r} = q \quad \text{for} \quad r = \frac{d}{2},$$

$$T = T_0 \quad \text{for} \quad x = 0.$$

We introduce the variables $X = x/d\mathrm{Pe}$, $r' = 2r/d$, $T' = T/T_0$, $q' = 2q/d\lambda$, and, omitting primes henceforth, we write the energy equation in the following form:

$$(1 - r^2)\frac{\partial T}{\partial X} = \frac{2}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{1}$$

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with boundary and initial conditions

$$\frac{\partial T}{\partial r} = 0 \quad \text{for} \quad r = 0, \tag{2a}$$

$$\frac{\partial T}{\partial r} = q \quad \text{for} \quad r = 1, \tag{2b}$$

$$T = 1 \quad \text{for} \quad X = 0. \tag{2c}$$

We solve Eq. (1) by a method which is a generalization of that presented in [1, 3] to the case of the flow of an incompressible liquid in a cylindrical channel. In doing this, we solve the equation for its highest derivative and then integrate it term-by-term with respect to the radius. After the integration we obtain:

$$r\frac{\partial T}{\partial r} = \frac{1}{2}\int_0^r \frac{\partial T}{\partial X}(1-r^2)\,r\,dr + C_1(X), \tag{3}$$

where $C_1(X) \equiv 0$, in view of the condition (2a). Next, we divide both sides of the resulting equation by r and again integrate it with respect to the radius:

$$T = T_e(X) + \frac{1}{2}\int_0^r \frac{1}{r}\int_0^r \frac{\partial T}{\partial X}(1-r^2)\,r\,dr\,dr. \tag{4}$$

Here $T_e(X)$ denotes the value of the temperature of the liquid at the axis of the tube. At a first approximation, we take the temperature profile in the form:

$$T = a(X) + b(X)\,r^{\alpha(X)}$$

or, after satisfying the condition (2b),

$$T = a + \frac{q}{\alpha}r^{\alpha}. \tag{5}$$

We substitute this approximate dependence of the temperature on the radius into the right side of Eq. (4) and make the two quadratures indicated. We then obtain a second approximation for the temperature profile in the form:

$$\begin{aligned} T = T_e(X) + \frac{1}{2}\Bigg\{&\frac{da}{dX}\left(\frac{r^2}{4}-\frac{r^4}{16}-\frac{7}{96}\right)+\left(\frac{1}{\alpha}\cdot\frac{dq}{dX}-\frac{q}{\alpha^2}\cdot\frac{d\alpha}{dX}\right)\\ &\times\left(\frac{r^{\alpha+2}}{(\alpha+2)^2}-\frac{r^{\alpha+4}}{(\alpha+4)^2}-\frac{32(2\alpha+7)}{(\alpha+2)^2(\alpha+4)^2(\alpha+6)(\alpha+8)}\right)\\ &+\frac{q}{\alpha}\cdot\frac{d\alpha}{dX}\Bigg[\frac{r^{\alpha+2}}{(\alpha+2)^2}\left(\ln r-\frac{2}{\alpha+2}\right)-\frac{r^{\alpha+4}}{(\alpha+4)^2}\left(\ln r-\frac{2}{\alpha+4}\right)\\ &-\frac{16}{(\alpha+4)^2(\alpha+6)}\left(\frac{2\alpha^2+25\alpha+76}{(\alpha+4)(\alpha+8)^2}-\frac{2\alpha^2+17\alpha+34}{(\alpha+2)^3}\right)\Bigg]\Bigg\}. \end{aligned} \tag{6}$$

In the method described in [1, 3], the second approximation for the desired function depends, in all, on only one unknown parameter, and, upon satisfying one of the boundary conditions, we obtain an ordinary differential equation, which enables us to find how this parameter varies with the longitudinal coordinate.

In our case, to find the functions $a(X)$ and $\alpha(X)$, we substitute the expression (5) into Eqs. (3) and (4) and we take the limits of integration in the integrals appearing in these equations equal to one. Then after integrating and making a number of simple manipulations, we obtain the two equations:

$$a = 1 + 8\int_0^X q\,dX - \frac{8q}{\alpha(\alpha+2)(\alpha+4)}; \tag{7a}$$

$$\frac{d\alpha}{dX} = \frac{(\alpha+2)^3(\alpha+4)^3}{9\alpha^2+58\alpha+96}\left[\frac{4-3\alpha}{\alpha}+\frac{3\alpha+10}{(\alpha+2)^2(\alpha+4)^2}\cdot\frac{1}{q}\cdot\frac{dq}{dX}\right] \tag{7b}$$

with the initial condition $\alpha \to \infty$ for $X \to 0$, which follows from the relation (2c). It is of interest to note that the expression (5) can, with the aid of the results obtained from solving Eqs. (7a) and (7b), describe the temperature profile accurate to within 8-10%, and it can sometimes be used by itself as an approximate solution of Eq. (1). In finding the unknown function $T_e(X)$ appearing in the expression for the temperature (4), we used the relationship for the bulk flow temperature. We write the mean temperature in the form:

$$\overline{T} = 4\int_0^1 (1-r^2)\,Tr\,dr. \tag{8}$$

Then from Eq. (1), using the condition (2b), we can easily obtain the following relationship:

$$4\int_0^1 (1-r^2)\frac{\partial T}{\partial X}r\,dr = 8\int_0^1 \frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)dr,$$
$$\overline{T} = 1 + 8\int_0^X q\,dX. \tag{9}$$

The constant of integration in the latter equation, by virtue of the condition (2c), is equal to one. Next, we substitute the expression (4) into Eq. (8), carry out the integration, and then, using the relation (9) between the resulting value of the bulk temperature and the heat flux, we find the unknown function $T_e(X)$:

$$T_c = \frac{1}{2}\left\{-\frac{7}{96}\cdot\frac{d\alpha}{dX} - \left(\frac{1}{\alpha}\cdot\frac{dq}{dX} - \frac{q}{\alpha^2}\cdot\frac{d\alpha}{dX}\right)\right.$$
$$\times\frac{32(2\alpha+7)}{(\alpha+2)^2(\alpha+4)^2(\alpha+6)(\alpha+8)} - \frac{16q}{\alpha(\alpha+4)^2(\alpha+6)}$$
$$\left.\times\left(\frac{2\alpha^2+25\alpha+76}{(\alpha+4)(\alpha+8)^2} - \frac{2\alpha^2+17\alpha+34}{(\alpha+2)^3}\right)\frac{d\alpha}{dX}\right\} + \overline{T}. \tag{10}$$

Thus the expression (6), together with the results of solving Eqs. (7a) and (7b) and also the expression (10), describes the temperature profile of the flow of an incompressible liquid moving in the initial portion of a circular tube.

For subsequent refinement of the temperature profile so obtained, we can use the expression (6) as a first approximation. It should be noted, in proceeding, that the steps to be followed remain unchanged, except that the second derivatives of the functions $a(X)$ and $\alpha(X)$, which arise in substituting the expression (6) into the right side of Eq. (4), can be obtained by differentiating Eqs. (7a) and (7b). However experiment shows that there is no practical need for further refinement of the expression (6), since the accuracy with which it describes the temperature profile (1.5%) is sufficient for engineering calculations. The dependence of the Nusselt number Nu on the coordinate X may then be represented in the following form:

$$\mathrm{Nu} = 48\Bigg/\left[11 - \frac{11\alpha+106}{(\alpha+2)(\alpha+4)(\alpha+6)(\alpha+8)}\cdot\frac{1}{q}\cdot\frac{dq}{dX}\right.$$
$$-\left(\frac{3\alpha-4}{\alpha} - \frac{1}{q}\cdot\frac{dq}{dX}\cdot\frac{3\alpha+10}{(\alpha+2)^2(\alpha+4)^2}\right)\times$$
$$\left.\times\frac{(\alpha+2)(\alpha+4)(33\alpha^4+864\alpha^3+7900\alpha^2+29680\alpha+38176)}{(9\alpha^2+58\alpha+96)(\alpha+6)^2(\alpha+8)^2}\right].$$

As an example of the application of our method we solved a number of heat-transfer problems differing from one another through the various laws describing the supply of heat along the wall of the tube. The change in the Nusselt number Nu with X is shown in Fig. 1a (curve 1, calculated for the case q = const). Curve 2, drawn for comparison, shows the variation of the Nusselt number Nu obtained as the result of numerically solving Eq. (1) [6]. For $X > 0.001$ the difference between the curves amounts to 1-2%. In Fig. 1b the dependence of the Nusselt number Nu on X is shown for an exponential law of heat supply ($q = \exp\cdot(cX)$). Curves 2-5 correspond to values of the coefficient c −10, −20, 20, and 40, respectively. Curve 1 corresponds to the heat supply law q = const.

By an analogous method we solve the problem of the friction of an incompressible liquid in laminar flow in the initial portion of a thermally insulated circular tube. We assume that the liquid has constant physical properties, that the pressure depends only on the longitudinal coordinate, and that the velocity profile is constant over the entrance section of the tube.

We introduce the notation:

$$X = \frac{x}{d\,\mathrm{Re}};\quad r' = \frac{2r}{d};\quad u' = \frac{u}{u_0};\quad v' = \frac{v}{u_0};\quad p' = \frac{P}{\rho u_0^2}$$

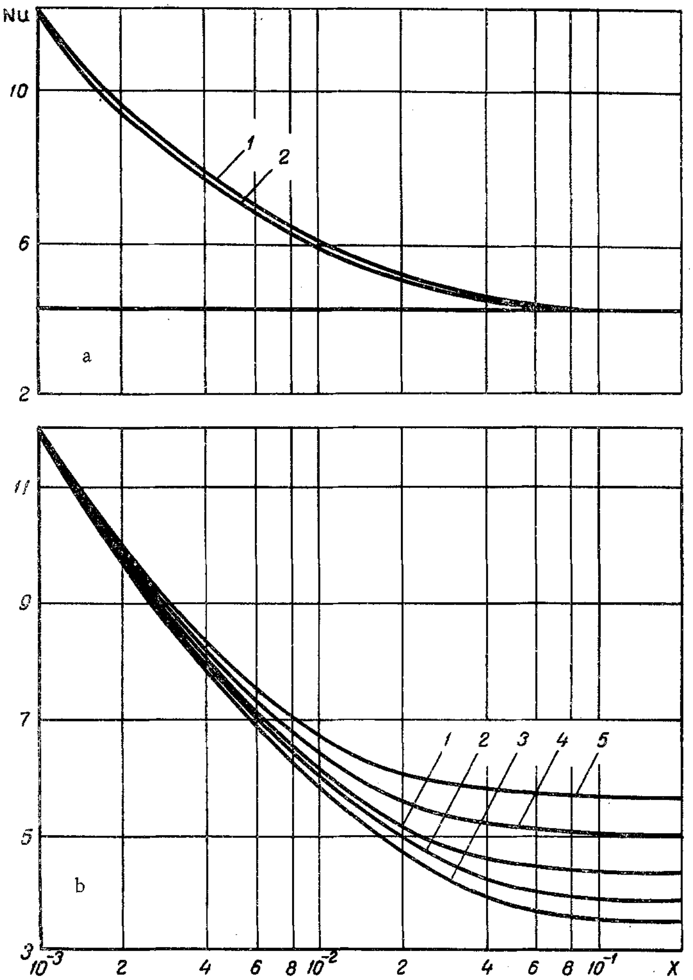

Fig. 1. Variation of the Nusselt number Nu along the length of a circular tube: a) q = const; b) q = exp (cX).

and, henceforth omitting primes for convenience, we write the system of equations describing the motion of the incompressible liquid in the tube in the following form:

$$u\,\frac{\partial u}{\partial X}+2v\,\mathrm{Re}\,\frac{\partial u}{\partial r}=-\frac{dP}{dX}+\frac{4}{r}\cdot\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right),$$
$$\frac{\partial}{\partial X}(ur)+2\mathrm{Re}\,\frac{\partial}{\partial r}(vr)=0 \qquad (11)$$

with the boundary and initial conditions:

$$r=0\quad \frac{\partial u}{\partial r}=0,\quad \frac{\partial v}{\partial r}=0, \qquad (12a)$$

$$r=1\quad u=0,\quad v=0, \qquad (12b)$$

$$X=0\quad u=1,\quad P=\frac{P_0}{\rho u_0^2},\quad v=0. \qquad (12c)$$

We now solve the second equation of the system (11) for v,

$$v=-\frac{1}{2\mathrm{Re}\,r}\int_0^r \frac{\partial u}{\partial X}\,rdr$$

and substitute it into the first equation of the same system. We solve the resulting equation for the highest derivative and, upon integrating twice, taking the conditions (12a) and (12b) into account, we obtain the following expression:

$$u=\frac{1}{4}\left[\frac{\partial}{\partial X}\int_0^r \frac{1}{r}\int_0^r u^2rdrdr-\frac{\partial}{\partial X}\int_0^1 \frac{1}{r}\int_0^r u^2rdrdr\right.$$

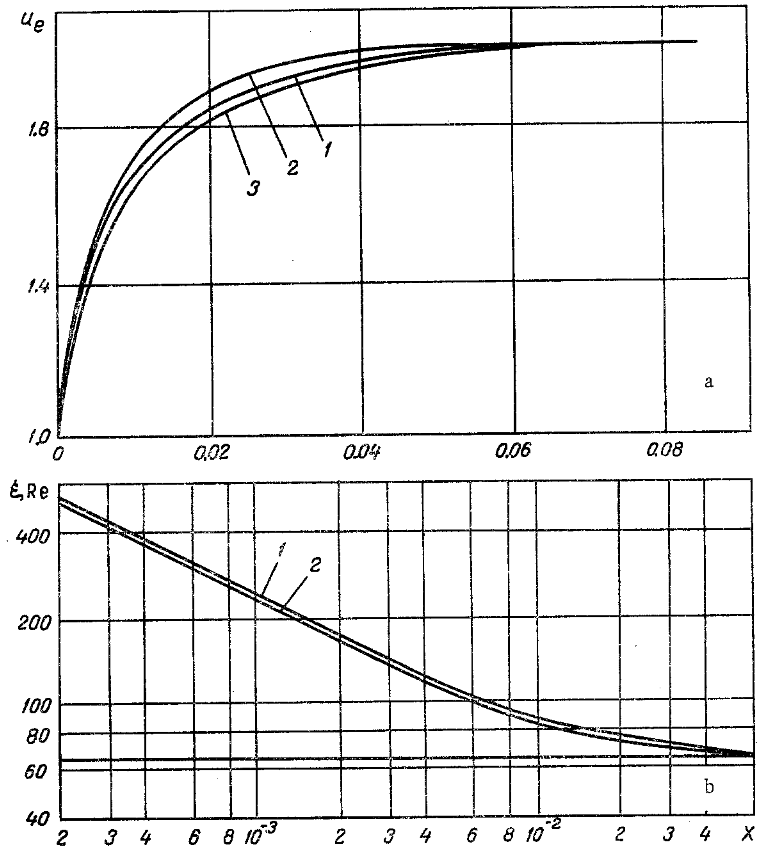

Fig. 2. Variation of the axial velocity in the initial portion of a circular tube (a) and of the local resistance coefficient in the initial portion of a circular tube (b).

$$-\int_0^r \frac{u}{r}\int_0^r \frac{\partial u}{\partial X}\, rdrdr + \int_0^1 \frac{u}{r}\int_0^r \frac{\partial u}{\partial X}\, rdrdr + \frac{r^2-1}{4}\cdot\frac{dP}{dX}\Bigg]. \tag{13}$$

We write the first approximation to the velocity profile in the form

$$u = u_e(X)\left(1 - r^{s(X)}\right) \tag{14}$$

and then substitute this expression into the right side of Eq. (13). Then, after integrating, we obtain a second approximation to the velocity profile:

$$u = \frac{d}{dX}\left[\frac{u_e^2}{4}\left(\frac{r^2-1}{4} - \frac{2\left(r^{s+2}-1\right)}{(s+2)^2} + \frac{r^{2s+2}-1}{4(s+1)^2}\right)\right]$$
$$-\frac{u_e}{4}\cdot\frac{du_e}{dX}\left(\frac{r^2-1}{4} - \frac{(s+4)\left(r^{s+2}-1\right)}{2(s+2)^2} + \frac{r^{2s+2}-1}{2(s+1)(s+2)}\right)$$
$$+\frac{u_e^2}{4}\,\frac{ds}{dX}\left[\frac{r^{s+2}}{(s+2)^2}\left(\ln r - \frac{2}{s+2}\right) - \frac{r^{2s+2}}{2(s+2)(s+1)}\right.$$
$$\left.\times\left(\ln r - \frac{3s+4}{2(s+2)(s+1)}\right) + \frac{s(5s+6)}{4(s+1)^2(s+2)^3}\right] + \frac{r^2-1}{16}\cdot\frac{dP}{dX}. \tag{15}$$

After eliminating v from the first equation of the system (11), we transform it to a form analogous to the equations (3) and (4):

$$\frac{\partial}{\partial X}\int_0^r u^2 rdr - u\int_0^r \frac{\partial u}{\partial X}\, rdr = -\frac{r^2}{2}\cdot\frac{dP}{dX} + 4r\frac{\partial u}{\partial r};$$

$$\frac{\partial}{\partial X}\int_0^r \frac{1}{r}\int_0^r u^2 rdrdr - \int_0^r \frac{u}{r}\int_0^r \frac{du}{\partial X}\, rdrdr = -\frac{r^2}{4}\cdot\frac{dP}{dX} + 4u + C_1.$$

We substitute expression (14) into the right side of the resulting equations and set the upper limits of the integrals appearing in these equations equal to one. Then, after making some simple manipulations, we obtain a system of two differential equations:

$$\frac{d}{dX}\left[u_e^2\frac{s^2}{2(s+1)(s+2)}\right]=-\frac{1}{2}\cdot\frac{dP}{dX}-4u_e s;$$

$$\frac{d}{dX}\left[u_e^2\left(\frac{1}{4}-\frac{2}{(s+2)^2}+\frac{1}{4(s+1)^2}\right)\right]$$

$$-u_e\frac{du_e}{dX}\left[\frac{1}{4}-\frac{s+4}{2(s+2)^2}+\frac{1}{2(s+1)(s+2)}\right]$$

$$+u_e^2\frac{ds}{dX}\left[-\frac{2}{(s+2)^3}+\frac{3s+4}{4(s+1)^2(s+2)^2}\right]=-\frac{1}{4}\cdot\frac{dP}{dX}-4u_e \tag{16}$$

with the initial conditions: for $X = 0$, $s \to \infty$; $u_e = 1$; $P = P_0/\rho u_0^2$ in view of the condition (12c).

To find yet another relationship between the unknown functions we use the outflow equation

$$2\int_0^1 urdr = 1.$$

Substituting the expression (14) into this latter equation, we obtain

$$u_e=\frac{s+2}{s}. \tag{17}$$

After some simple manipulations the system (16) reduces to

$$\frac{dP}{dX}=\frac{1}{(s+1)^2}\cdot\frac{ds}{dX}-8(s+2);$$

$$\frac{ds}{dX}=4(s+2)\Big/\left[\frac{1}{s^2}\left(1-\frac{2(3s+4)}{(s+2)^2}+\frac{3s+2}{2(s+1)^2}\right)-\frac{(3s+5)s}{2(s+1)^3(s+2)}\right]. \tag{18}$$

Thus the expression (15), with the aid of the relation (17) and the results obtained from solving the system (18), describes the velocity profile for the flow of an incompressible liquid in a cylindrical tube. Just as in the heat-transfer case, the accuracy of the resulting expression is in practice sufficient for engineering calculations.

The dependence of the resistance coefficient on X may be represented as follows:

$$\xi\,\mathrm{Re}=16(s+2)-8\Big/\left[\frac{(s+1)^2}{s^2(s-2)}\left(1-\frac{(3s+4)}{(s+2)^2}+\frac{3s+2}{2(s+1)^2}\right)-\frac{(3s+5)s}{2(s+1)(s^2-4)}\right].$$

In Fig. 2a the variation of the axial velocity with X is shown (curve 1); as a comparison, curves 2 and 3 show the variation of the axial velocity as computed by the methods of [2] and [4], which are, as is well known, among the most accurate.

In Fig. 2b the dependence of ξ Re on X is shown (curve 1); as a comparison, curve 2 represents Targ's solution [2]. For $X > 0.001$, the difference among these solutions amounts to 1-2%.

It should be pointed out, in conclusion, that the practical application of our method is closely connected with the choice of a good initial approximation. This approximation should reflect the main features of the desired function and, if possible, should approach its asymptotic value.

NOTATION

x	is the axial coordinate;
r	is the radial coordinate;
u, v	are the axial and radial components of the velocity;
ρ	is the density of the liquid;
P	is the pressure of the liquid;
T	is the temperature of the liquid;

q	is the heat flux at the walls of the tube;
P_0	is the pressure of the liquid at the tube entrance;
u_0	is the axial velocity component at the tube entrance;
T_0	is the temperature of the liquid at the tube entrance;
μ_0	is the liquid viscosity at the tube entrance;
c_p	is the specific heat of the liquid;
$Re = du_0/\mu_0$	is the Reynolds number;
Pe	is the Peclet number;
ξ	is the coefficient of local friction of the liquid;
Nu	is the Nusselt number;
d	is the diameter of the tube;
$\bar{u}$	is the mean liquid velocity;
λ	is the thermal conductivity of the liquid.

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